Triangle-Free Strongly Circular-Perfect Graphs

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Abstract

Zhu [15] introduced circular-perfect graphs as a superclass of the well-known perfect graphs and as an important χ-bound class of graphs with the smallest non-trivial χ-binding function $\chi(G) \leq \omega(G)+1$. Perfect graphs have been recently characterized as those graphs without odd holes and odd antiholes as induced subgraphs [4]; in particular, perfect graphs are closed under complementation [7]. In contrary, circular-perfect graphs are not closed under complementation and the list of forbidden subgraphs is unknown.

We study strongly circular-perfect graphs: a circular-perfect graph is strongly circular-perfect if its complement is circular-perfect as well. This subclass entails perfect graphs, odd holes, and odd antiholes. As main result, we fully characterize the triangle-free strongly circular-perfect graphs, and prove that, for this graph class, both the stable set problem and the recognition problem can be solved in polynomial time.

Moreover, we address the characterization of strongly circular-perfect graphs by means of forbidden subgraphs. Results from [9] suggest that formulating a corresponding conjecture for circular-perfect graphs is difficult; it is even unknown which triangle-free graphs are minimal circular-imperfect. We present the complete list of all triangle-free minimal not strongly circular-perfect graphs.

1 Introduction

Coloring the vertices of a graph is an important concept with a large variety of applications. Let $G = (V, E)$ be a graph with vertex set V and edge set E, then a k-*coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* $\chi(G)$; calculating $\chi(G)$ is NP-hard in general. In a set of k pairwise adjacent vertices, called *clique* K_k, all k vertices have to be colored differently. Thus the size of a largest clique in G, the *clique number* $\omega(G)$, is a trivial lower bound on $\chi(G)$; this bound is hard to evaluate as well.

Berge [2] proposed to call a graph G *perfect* if each induced subgraph $G' \subseteq G$ admits an $\omega(G')$-coloring. Perfect graphs have been recently characterized as those graphs without chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C}_{2k+1}$, the *odd antiholes*, as induced subgraphs (Strong Perfect Graph Theorem [4]). (The *complement* $\overline{G}$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.) In particular, the class of perfect graphs is closed under complementation [7]. Perfect graphs turned out to be an interesting and important class with a rich structure, see [10] for a recent survey. For instance, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [5].

1.1 Strongly circular-perfect graphs

As a generalization of perfect graphs, Zhu [15] introduced recently the class of circular-perfect graphs based on the following more general coloring concept. For integers $k \geq 2d$, a (k, d)-*circular coloring* of a graph $G = (V, E)$ with at least one edge is a mapping $f : V \to \{0, \ldots, k-1\}$ with $|f(u) - f(v)| \geq d \bmod k$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G; we have $\chi_c(G) \leq \chi(G)$ since every $(k, 1)$-circular coloring is a usual k-coloring of G. (Note that $\chi_c(G)$ is sometimes called the star chromatic number [3,13].) The circular chromatic number of a stable set is set to be 1.

In order to obtain a lower bound on $\chi_c(G)$, we generalize cliques as follows: Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \ldots, k-1$ and edges ij iff $d \leq |i - j| \leq k - d$. Such graphs $K_{k/d}$ are called *circular cliques* (or sometimes antiwebs [11,14]) and are said to be *prime* if $\gcd(k, d) = 1$. Circular cliques include all cliques $K_t = K_{t/1}$, all odd antiholes $\overline{C}_{2t+1} = K_{(2t+1)/2}$, and all odd holes $C_{2t+1} = K_{(2t+1)/t}$, see Figure 1. The *circular clique number* is defined as $\omega_c(G) =$

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$\max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k,d) = 1\}$, and we immediately obtain that $\omega(G) \leq \omega_c(G)$. (Note: in this paper, we always denote an induced subgraph G' of G by $G' \subseteq G$.)

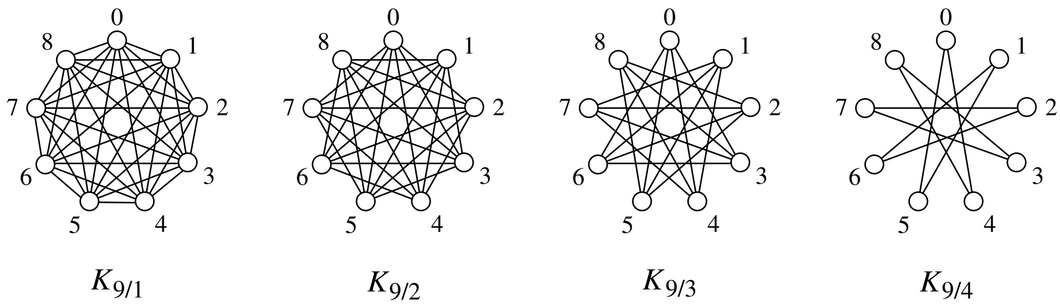


Fig. 1. The circular cliques on nine vertices

Every circular clique $K_{k/d}$ clearly admits a (k,d)-circular coloring (simply take the vertex numbers as colors, as in Figure 1), but no (k',d')-circular coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [3]. Thus we obtain, for any graph G, the following chain of inequalities:

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{1}$$

A graph G is called *circular-perfect* if, for each induced subgraph $G' \subseteq G$, circular clique number $\omega_c(G')$ and circular chromatic number $\chi_c(G')$ coincide. Obviously, every perfect graph has this property by (1) as $\omega(G')$ equals $\chi(G')$. Moreover, any circular clique is circular-perfect as well [15,1]. Thus circular-perfect graphs constitute a proper superclass of perfect graphs.

Another natural extension of perfect graphs was introduced by Gyárfás [6] as follows: A family $\mathcal{G}$ of graphs is called χ-bound with χ-binding function b if $\chi(G') \leq b(\omega(G'))$ holds for all induced subgraphs G' of $G \in \mathcal{G}$. Thus, this concept uses functions in $\omega(G)$ as *upper* bound on $\chi(G)$. Since it is known for any graph G that $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [15] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [13], we obtain that circular perfect graphs G satisfy the following Vizing-like property

$$\omega(G) \leq \chi(G) \leq \omega(G) + 1. \tag{2}$$

Thus, the class of circular-perfect graphs is χ-bound with the smallest non-trivial χ-binding function. In particular, this χ-binding function is best possible for a *proper* superclass of perfect graphs implying that circular-perfect graphs admit coloring properties almost as nice as perfect graphs. In contrary to perfect graphs, circular-perfect graphs are not closed under complementation and the list of forbidden subgraphs is unknown.

In this paper, we study *strongly circular-perfect graphs*: a circular-perfect graph is strongly circular-perfect if its complement is circular-perfect as well. We address the problem of finding the minimal not strongly circular-perfect graphs and provide complete answers in the triangle-free case.

1.2 Summary of results

We first address the problem which circular cliques occur in strongly circular-perfect graphs, see Section 2. For that we fully characterize which circular cliques have a circular-perfect complement (Theorem 3).

Section 3 deals with triangle-free strongly circular-perfect graphs. A graph G is said to be an *interlaced odd hole* if and only if the vertex set of G admits a suitable partition $((A_i)_{1\leq i\leq 2p+1}, (B_i)_{1\leq i\leq 2p+1})$ into $2p+1$ (with $p \geq 2$) non-empty sets $A_1, \ldots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \ldots, B_{2p+1}$ such that

(1) $\forall 1 \leq i \leq 2p+1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (indices modulo $2p+1$),
(2) $\forall 1 \leq i \leq 2p+1$, $B_i \neq \emptyset$ implies $|A_i| = 1$,

and the edge set of G is equal to $\cup_{i=1,\ldots,2p+1}(E_i \cup E'_i)$, where E_i (resp. E'_i) denotes the set of all edges between A_i and A_{i+1} (resp. between A_i and B_i); see Figure 1.2 for an example (the sets of vertices in B_i are grey).

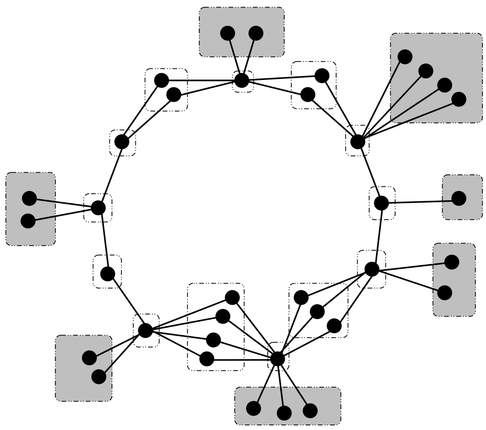


Fig. 2. An interlaced odd hole

We prove that a graph G is triangle-free strongly circular-perfect if and only if G is bipartite or an interlaced odd hole (Theorem 15). We use this characterization of triangle-free strongly circular-perfect graphs to exhibit that both the stable set problem and the recognition problem can be solved in polynomial time for such graphs (see Theorem 15 and Algorithm 1).

In Section 4, we finally address, motivated by the Strong Perfect Graph Theorem, the problem of finding all forbidden subgraphs for the class of strongly circular-perfect graphs. Results in [9] indicate that even formulating an appropriate conjecture for circular-perfect graphs is difficult, e.g., it is unknown which triangle-free graphs are not circular-perfect. We present the complete list of all triangle-free graphs which are minimal not strongly circular-perfect (Theorem 22).

2 Circular cliques in strongly circular-perfect graphs

In this section, we solve the problem which prime circular cliques occur as induced subgraphs of a strongly circular-perfect graph. As the class of strongly circular-perfect graphs is closed under complementation, this is equivalent to ask which circular cliques have a circular-perfect complement.

The complement of a circular clique is called a *web* and we denote by $C_n^{\omega-1}$ the web $\overline{K_{n/\omega}}$, that is the graph with vertices $0, \ldots, n-1$ and edges ij such that i and j differ by at most $\omega - 1 \pmod n$, and $i \neq j$. In particular, the maximum clique size of $C_n^{\omega-1}$ is ω.

For that, we use the following result on claw-free graphs (note that webs are claw-free as the neighborhood of any node splits into two cliques).

Lemma 1 *[9] A claw-free graph does not contain any prime antiwebs different from cliques, odd antiholes, and odd holes.*

This immediately implies for circular clique numbers of claw-free graphs:

Corollary 2 *Let G be a claw-free graph.*

(1) If $\omega(G) = 2$, then $\omega_c(G) = 2$ follows iff G is perfect and $\omega_c(G) = 2 + \frac{1}{k}$ iff G is imperfect and C_{2k+1} is the shortest odd hole in G.
(2) If $\omega(G) \geq 3$, then $\omega_c(G) = \max\{\omega(G), k' + \frac{1}{2}\}$ where $\overline{C}_{2k'+1}$ is the longest odd antihole in G.

This enables us to completely characterize the circular-(im)perfection of webs as follows (note that the proof of assertion (3) is given in [9]).

Theorem 3 *The web C_n^k is*

(1) circular-perfect if $k = 1$ or $n \leq 2(k+1)+1$,
(2) circular-perfect if $k = 2$ and $n \equiv 0 \pmod 3$,
(3) minimal circular-imperfect if $k = 2$ and $n \equiv 1 \pmod 3$,
(4) circular-imperfect if $k = 2$ and $n \equiv 2 \pmod 3$,
(5) circular-imperfect if $k \geq 3$ and $n \geq 2(k+2)$.

Proof. For that, we prove the following sequence of claims.

Claim 4 *Any web C_n^k with $k = 1$ or $n \leq 2(k+1)+1$ is circular-perfect.*

The webs C_n^1 are obviously all circular-perfect. Moreover, C_n^k is perfect if $n \leq 2(k+1)$ and an odd antihole if $n = 2(k+1)+1$, thus C_n^k is circular-perfect if $n \leq 2(k+1)+1$. $\diamond$

Thus Claim 4 verifies already assertion (1). In the sequel, we have to consider webs C_n^k with $k \geq 2$ and $n \geq 2(k+2)$ only. In [9] it is shown that the webs $C_{3\alpha+1}^2$ are minimal circular-imperfect for $\alpha \geq 3$; this already ensures assertion (3). In order to show circular-perfection for the webs $C_{3\alpha}^2$ with $\alpha \geq 3$ and circular-imperfection for all remaining webs, we need the following.

Claim 5 *C_n^k with $k \geq 2$, $n \geq 2(k+2)$ is circular-perfect only if $\omega(C_n^k) = \chi(C_n^k)$.*

We have $\omega(C_n^k) \geq 3$ and Corollary 2(2) implies $\omega_c(C_n^k) = \max\{k+1, k'+\frac{1}{2}\}$ taken over all odd antiholes $C_{2k'+1}^{k'-1}$ in C_n^k. As $C_{n'}^l \subset C_n^k$ holds only if $l < k$ due to Trotter [12], we obtain that $k+1 > k'+\frac{1}{2}$ for any odd antihole $C_{2k'+1}^{k'-1}$ in C_n^k. Thus, $\omega(C_n^k) = k+1 = \omega_c(C_n^k)$ holds, implying the assertion by $\lceil \chi_c(C_n^k) \rceil = \chi(C_n^k)$. $\diamond$

Claim 6 *For a web C_n^k with $n \geq 2(k+2)$, we have $\omega(C_n^k) < \chi(C_n^k)$ if and only if $(k+1) \not| \, n$.*

For any non-complete web C_n^k, it is well-known that $\chi(C_n^k) = \lceil \frac{n}{\alpha} \rceil$ holds where $\alpha = \alpha(C_n^k) = \lfloor \frac{n}{k+1} \rfloor$. Assuming $n = \alpha(k+1)+r$ with $r < k+1$ we obtain

$$\chi(C_n^k) = \left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{\alpha(k+1)+r}{\alpha} \right\rceil = k+1+\left\lceil \frac{r}{\alpha} \right\rceil$$

implying $k+1 = \omega(C_n^k) < \chi(C_n^k)$ whenever $r > 0$, i.e., whenever $(k+1) \not| \, n$. $\diamond$

Combining Claim 5 and Claim 6 proves assertion (4); the only possible circular-perfect webs C_n^k satisfy $(k+1)|n$. This is obviously true for the webs $C_{3\alpha}^2$. In order to show their circular-perfection, we have to ensure that none of them contains a minimal circular-imperfect induced subgraph. By $\omega(C_{3\alpha}^2) = 3 = \chi(C_{3\alpha}^2)$, every induced subgraph G' of $C_{3\alpha}^2$ is clearly 3-colorable. Thus, $\omega(G') = 3$ implies $\omega_c(G') = \chi_c(G')$. The next claim also excludes the occurrence of minimal circular-imperfect induced subgraphs with less clique number:

Claim 7 *No web C_n^2 contains a (minimal) circular-imperfect graph with clique number 2 as induced subgraph.*

Suppose $G' \subset C_n^2$ is triangle-free. Then G' does not admit any vertex of degree 3 (since every vertex of C_n^2 together with three of its neighbors contains a triangle). The assertion follows since all graphs with maximal degree 2 are collections of paths and cycles, and are thus circular-perfect. $\diamond$

Hence, assertion (2) is true. For the last assertion (5), it is left to show that every web C_n^k with $k \geq 3$ and $(k+1)|n$ contains a circular-imperfect induced subgraph.

Claim 8 *Any web $C^k_{\alpha(k+1)}$ with $k, \alpha \geq 3$ is circular-imperfect.*

We show that all those webs $C^k_{\alpha(k+1)}$ contain a circular-imperfect web as induced subgraph. Claim 6 implies that $C^{k-1}_{\alpha k-1}$ is circular-imperfect as $k \not| \ (\alpha k - 1)$. We show $C^2_{3\alpha-1} \subseteq C^k_{\alpha(k+1)}$ if $\alpha < k$ and $C^{k-1}_{\alpha k-1} \subseteq C^k_{\alpha(k+1)}$ if $\alpha \geq k$ with the help of the following result of Trotter [12]:

$$C^{k'}_{n'} \subseteq C^k_n \text{ if and only if } \frac{k'}{k} n \leq n' \leq \frac{k'+1}{k+1} n$$

Hence, we have $C^2_{3\alpha-1} \subseteq C^k_{\alpha(k+1)}$ for $\alpha < k$ since

$$\frac{2}{k}\alpha(k+1) = 2\alpha + \frac{2\alpha}{k} \leq 3\alpha - 1 \leq \frac{3}{k+1}\alpha(k+1) = 3\alpha$$

holds: the first inequality is satisfied by $2\frac{\alpha}{k} < 2 \leq \alpha - 1$ if $\alpha < k$ and $\alpha \geq 3$; the second one is trivial. Moreover, $C^{k-1}_{\alpha k-1} \subseteq C^k_{\alpha(k+1)}$ follows for $\alpha \geq k$ since

$$\frac{k-1}{k}\alpha(k+1) = \alpha(k-1) + \frac{\alpha(k-1)}{k} \leq \alpha k - 1 \leq \frac{k}{k+1}\alpha(k+1) = \alpha k$$

holds: the first inequality is satisfied since $\frac{\alpha(k-1)}{k} \leq \alpha - 1$ is true due to $\alpha \geq k$; the second inequality obviously holds again. $\diamond$

Thus, a web C^k_n with $k \geq 3$ and $n > 2(k+1)+1$ is circular-imperfect: if $(k+1) \not| \ n$ by Claim 6 and if $(k+1)|n$ by Claim 8, finally verifying assertion (5). $\Box$

Corollary 9 *The induced prime circular cliques of a strongly circular-perfect graph are cliques, odd antiholes and odd holes.*

Corollary 10 *A circular clique is strongly circular-perfect if and only if it is a clique, an odd antihole, an odd hole, a stable set, or of the form $K_{3k/3}$ with $k \geq 3$.*

We end this section with two lemmas discussing the adjacency of odd (anti)holes in strongly circular-perfect graphs and the behaviour under multiplying vertices. We call an induced subgraph $G' \subseteq G$ *dominating* (resp. *antidominating*) if every vertex in $G - G'$ has at least one neighbor (resp. non-neighbor) in G'.

Lemma 11 *Every odd hole or odd antihole in a strongly circular-perfect graph is dominating as well as antidominating.*

Proof. We know from [9] that no vertex of a circular-perfect graph G is totally joined to any odd hole or odd antihole C in G, thus C is antidominating. If G is strongly circular-perfect, then the same applies to $\overline{G}$ and C is also dominating. $\Box$

Let $G_{v,S}$ be the graph obtained by multiplication of a vertex v in G by a stable set S (i.e., v is replaced by $|S|$ vertices having exactly the same neighbors as v in G) and let G_{v+w} be the graph obtained by adding a node w to G, whose only neighbour is v.

Lemma 12

(i) $G_{v,S}$ *is circular-perfect if and only if* G *is circular-perfect;*
(ii) G_{v+w} *is circular-perfect if and only if* G *is circular-perfect.*

Proof. Notice that both graphs $G_{v,S}$ and G_{v+w} contain G as an induced subgraph, so we only have to prove the *if* part of both assertions. Hence assume that G is circular-perfect.

The $|S|$ copies of the vertex v in $G_{v,S}$ are pairwise non-adjacent and have the same neighbors. Thus, $G_{v,S}$ cannot contain any new circular cliques and $\omega_c(G_{v,S}) = \omega_c(G)$ follows. Furthermore, all copies of v can receive the same color, namely the previous color of v, implying $\chi_c(G_{v,S}) = \chi_c(G)$. The same is obviously true for all induced subgraphs. Hence, as multiplication of vertices does neither change the circular clique nor the circular chromatic number, the graph $G_{v,S}$ is circular-perfect.

If G is a stable set then G_{v+w} is perfect and therefore circular-perfect. If G is not a stable set then adding the leaf w does neither change the circular clique number nor the circular chromatic number. Therefore G_{v+w} is circular-perfect. $\Box$

3 Triangle-free strongly circular-perfect graphs

The aim of this section is to fully characterize the triangle-free strongly circular-perfect graphs and to address stable set and recognition problem for these graphs.

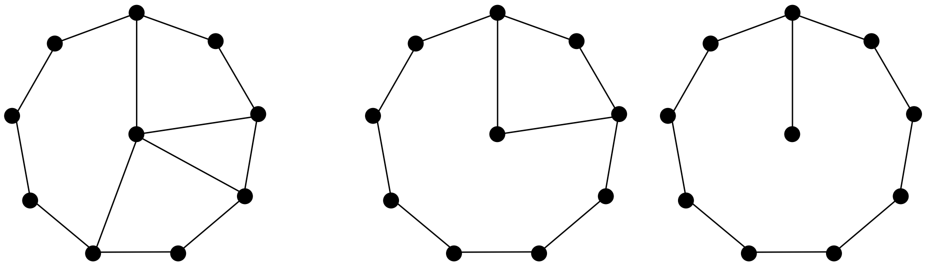
Corollary 9 implies that the only prime circular cliques in a triangle-free strongly circular-perfect graph are cliques and odd holes; we first consider shortest odd holes in triangle-free strongly circular-perfect graphs.

Lemma 13 *Every vertex outside a shortest odd hole* $\mathcal{O}$ *of a triangle-free graph has at most two neighbours in* $\mathcal{O}$*. Furthermore, if* x *has two such neighbours* y_1 *and* y_2 *then* y_2 *has a common neighbour with* y_1 *in* $\mathcal{O}$*.*

Proof. Let x be a vertex outside a shortest odd hole $\mathcal{O}$. W.l.o.g. assume that the vertices of $\mathcal{O}$ are labelled in the canonical cyclic order as $\{1, \ldots, 2p+1\}$ and let $x_1 < \ldots < x_k$ be the neighbours of x in $\mathcal{O}$. For every $2 \le i \le k$, let $r_i = x_i - x_{i-1} - 1$ and let $r_1 = x_1 + 2p + 1 - x_k - 1$ (see Fig. 2a). We have

$$2p + 1 = |\mathcal{O}| = k + \sum_{i=1,\ldots,k} r_i \tag{3}$$

Since G is triangle-free, we have $r_i > 0, \forall 1 \leq i \leq k$. As $|\mathcal{O}| = 2p + 1$ is odd, Eq. (3) implies that there exists j such that r_j is even. As $\mathcal{O}$ is a shortest odd hole, this implies that $r_j = |\mathcal{O}| - 1$ or $r_j = |\mathcal{O}| - 3$. As all r_i are positive, Eq. (3) implies $k = 1$ (resp. $k = 2$) if $r_j = |\mathcal{O}| - 1$ (see Fig. 2c) (resp. $r_j = |\mathcal{O}| - 3$ (see Fig. 2b)). □



$r_1 = 1,\ r_2 = 0,\ r_3 = 1,\ r_4 = 3$ $\qquad$ $r_1 = 1,\ r_2 = 6$ $\qquad$ $r_1 = 8$

Fig. 2a $\qquad$ Fig. 2b $\qquad$ Fig. 2c

Lemma 14 *Let G be a strongly circular-perfect graph with a shortest odd hole $\mathcal{O}$. Then every edge is incident to the odd hole $\mathcal{O}$.*

Proof. Suppose that there is an edge xy which is not incident to $\mathcal{O}$. Let $2p+1$ be the size of $\mathcal{O}$. Then the subgraph H induced by $\mathcal{O}$ and the vertices x and y is a strongly circular-perfect graph, with stability number at most $p + 1$. Since x has at most 2 neighbours in $\mathcal{O}$, the vertex x does not see at least one maximum stable set of $\mathcal{O}$. Thus H has stability number $p + 1$. Due to Theorem 3, this implies that the circular clique number of $\overline{H}$ is $p + 1$. As $\overline{H}$ is circular-perfect, we have $\chi_c(\overline{H}) = p + 1$. Since $\chi(\overline{H})$ is the upper integer part of $\chi_c(\overline{H})$, the graph $\overline{H}$ is $(p + 1)$-colorable. Hence H admits a covering with at most $p + 1$ cliques $Q_1, \ldots Q_{p+1}$. Let Q_x (resp. Q_y) be the clique containing x (resp. y). Then at least one of Q_x and Q_y meets $\mathcal{O}$ in two consecutive vertices, and has therefore at least 3 vertices. This implies that one of x and y belongs to a triangle: a contradiction. □

We are now prepared to prove the following characterization:

Theorem 15 *A triangle-free graph G is strongly circular-perfect if and only if G is bipartite or an interlaced odd hole.*

Proof. ***Only if.*** Let G be a triangle-free strongly circular-perfect graph. If G is perfect then G is bipartite and we have nothing to prove. If G is not perfect, then G contains an induced odd hole or antihole by the Strong Perfect Graph Theorem. Since G is triangle-free, this means that G contains at least one induced odd hole $\mathcal{O}$. Let $2k + 1$ be the size of this shortest odd hole.

The proof is by induction on the number of vertices: let $H(p, n)$ be the hypothesis "Every triangle-free strongly circular-perfect graph with a shortest odd hole of size

$2p+1$ and at most n vertices is an interlaced odd hole".

Let n be the number of vertices of G: we have $n \geq 2p+1$. $H(p, 2p+1)$ is obviously true, hence assume that $n > 2p+1$ and that $H(p, n-1)$ is true.

There exists a vertex x outside the shortest odd hole $\mathcal{O}$. By induction hypothesis, $G - x$ is an interlaced odd hole and there exists a suitable partition of $G - x$ into $2p+1$ non-empty sets $A_1, \ldots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \ldots, B_{2p+1}$, i.e.,

(1) $\forall 1 \leq i \leq 2p+1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (with indices modulo $2p+1$),
(2) $\forall 1 \leq i \leq 2p+1$, $B_i \neq \emptyset$ implies $|A_i| = 1$,

and the edge set of $G - x$ is equal to $\cup_{i=1,\ldots,2p+1}(E_i \cup E'_i)$, where E_i (resp. E'_i) denotes the set of all edges between A_i and A_{i+1} (resp. between A_i and B_i).

By Lemma 14 and Lemma 13, x is of degree 1 or 2.

If x is of degree 1 then the neighbour y of x belongs to $\mathcal{O}$ due to Lemma 14 again. Since y belongs to an odd hole of $G - x$, there exits a set A_j such that $y \in A_j$. For every $1 \leq i \leq 2p+1$ with $i \neq j$, let $B'_i = B_i$ and let $B'_j = B_j \cup \{x\}$. Then obviously $A_1, \ldots, A_{2p+1}$ and $B'_1, \ldots, B'_{2p+1}$ is a suitable partition of G. Thus G is an interlaced odd hole.

If x is of degree 2 then the neighbours y_1 and y_2 of x belong to $\mathcal{O}$ due to Lemma 14. By Lemma 13, there exists an index j such that y_1 belongs to A_{j-1} and y_2 belongs to A_{j+1} (or vice-versa). If A_{j-1} has at least two vertices, then there exists a shortest odd hole such that xy_1 is not incident to it, a contradiction to Lemma 14. Hence $|A_{j-1}| = |A_{j+1}| = 1$. Let $\mathcal{O}'$ be the shortest odd hole $(\mathcal{O} \cup x) \setminus A_j$. If $B_j \neq \emptyset$ then there are no edges between B_j and $\mathcal{O}'$: a contradiction to Lemma 11 . Thus $B_j = \emptyset$.

For every $1 \leq i \leq 2p+1$ with $i \neq j$, let $A'_i = A_i$ and let $A'_j = A_j \cup \{x\}$. Then obviously $A'_1, \ldots, A'_{2p+1}$ and $B_1, \ldots, B_{2p+1}$ is a suitable partition of G. Thus G is an interlaced odd hole. $\diamond$

If.

Let G be a bipartite graph or an interlaced odd hole. If G is bipartite then G is perfect and therefore strongly circular-perfect. If G is an interlaced odd hole, then G is circular-perfect due to Lemma 12.

It remains to show that $\overline{G}$ is circular-perfect as well. The proof is by contradiction: assume that $\overline{G}$ is not circular-perfect and take an induced subgraph H of G such that $\overline{H}$ is a minimal circular-imperfect graph. We have $\omega_c(\overline{H}) < \chi_c(\overline{H})$.

Notice that H is not perfect as $\overline{H}$ is circular-imperfect. Since H is an induced subgraph of G this implies that H is an interlaced odd hole and is not an odd hole. H admits a suitable partition into $2p+1$ non-empty sets $A_1, \ldots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \ldots, B_{2p+1}$.

Claim 16 *We have* $\omega_c(\overline{H}) = \alpha(H)$.

By construction, $2p+1$ is the size of every odd hole of H. As H is triangle-free, $\overline{H}$ is claw-free and the prime induced circular-cliques of $\overline{H}$ are stable sets, cliques, odd holes and odd antiholes due to Lemma 1. Thus $\omega_c(\overline{H}) = \max\{p + 1/2, \omega(\overline{H}) = \alpha(H)\}$. As H is not an odd hole, there exists a set A_i with at least 2 vertices or a non-empty set B_i, and in both cases, $\alpha(H) \geq p+1$. Therefore $\omega_c(\overline{H}) = \alpha(H)$ as required. $\diamond$

Claim 17 *H does not have any vertex of degree 1.*

Assume that H has a vertex x of degree 1 and let y be the neighbour of x: the removal of y yields a bipartite graph. Hence $H \setminus \{x, y\}$ has a covering $\mathcal{Q}$ with $\alpha(H \setminus \{x, y\})$ cliques. Notice that if S is any maximum stable set of $H \setminus \{x, y\}$ then $S \cup \{x\}$ is a stable set of H. Hence $\alpha(H) > \alpha(H \setminus \{x, y\})$. Therefore $\mathcal{Q} \cup \{\{x, y\}\}$ is a covering with at most $\alpha(H)$ cliques of H. Thus

$$\alpha(H) = \omega_c(\overline{H}) \leq \chi_c(\overline{H}) \leq \chi(\overline{H}) \leq \alpha(H)$$

yields $\omega_c(\overline{H}) = \chi_c(\overline{H})$, a contradiction. $\diamond$

Claim 18 *For every $1 \leq i \leq 2p+1$, the set A_i is a singleton.*

The proof is similar to the proof of Claim 17. Assume that there is a set Q_i with at least two vertices $\{x, x'\}$. The vertex x has two neighbours y and z. The removal of the set of vertices $\{x, y, z\}$ yields a bipartite graph. Hence $H \setminus \{x, y, z\}$ has a covering $\mathcal{Q}$ with $\alpha(H \setminus \{x, y, z\})$ cliques. We have $\alpha(H) > \alpha(H \setminus \{x, y, z\})$.

Notice that x' is isolated in $H \setminus \{x, y, z\}$. Hence $\{x'\} \in \mathcal{Q}$. Thus $(\mathcal{Q} \setminus \{\{x'\}\}) \cup \{\{x', y\}, \{x, z\}\}$ is a covering with at most $\alpha(H)$ cliques of H. Since $H - x$ is strongly circular-perfect, this implies that H is strongly circular-perfect, a contradiction. $\diamond$

Therefore, every set B_i is empty due to Claim 17 and every set A_i is a singleton due to Claim 18. Thus H is an odd hole, a final contradiction. $\square$

In order to treat the stable set problem for triangle-free strongly circular-perfect graphs, we show that they belong to a subclass of the well-known t-perfect graphs for which a maximum weight stable set can be found in polynomial time [5]. A graph is *almost-bipartite* if it has a vertex v such that $G - v$ is bipartite; such graphs are t-perfect (see [5]).

Lemma 19 *Interlaced odd holes are almost-bipartite.*

Proof. Let G be an interlaced odd hole and $((A_i)_{1\leq i\leq 2p+1}, (B_i)_{1\leq i\leq 2p+1})$ be a suitable partition of G. Obviously at least one of the sets A_i is a singleton $\{v\}$ and $G - v$ is bipartite, as v belongs to all odd holes of G. □

As bipartite graphs are almost-bipartite, Lemma 19 and Theorem 15 imply:

Corollary 20 *In a triangle-free strongly circular-perfect graph, a maximum weight stable set can be found in polynomial time.*

Remark. Interlaced odd holes are also near-bipartite (for every vertex v, $G - N(v)$ is bipartite), nearly-bipartite planar (a planar graph such that at most two faces are bounded by an odd number of edges), series-parallel (it does not contain a subdivision of K_4), strongly t-perfect (it does not contain a subdivision of K_4 such that all four circuits corresponding to triangles in K_4 are odd).

It is an open question whether there exists a polynomial time algorithm to recognize strongly circular-perfect graphs (resp. circular-perfect graphs). However, it is easy to derive such an algorithm for *triangle-free* strongly circular-perfect graphs from our characterization (see Algorithm 1).

Theorem 21 *Algorithm 1 works correct in polynomial time.*

Sketch of the proof.

1-3 Recognizing a bipartite graph in polynomial time is a standard exercise.

4 The graph is not bipartite. If it is triangle-free without an odd hole then it is perfect, and therefore bipartite, a contradiction. Hence the graph has a triangle or a shortest odd hole. In both cases, there exists a shortest odd cycle $\mathcal{O}$ which can be exhibited in polynomial time [8].

5-7 If a shortest odd cycle has size 3 then the graph is not triangle-free.

8-11 The graph is triangle-free. With every vertex o_i of the shortest odd hole $\mathcal{O}$, we define the set B_i as the set of neighbours of o_i of degree 1, and A_i as the union of o_i and vertices of degree two with neighbours o_{i-1} and o_{i+1}.

12- Notice that if the graph is an interlaced odd hole, then the sets A_i and B_i should be a suitable partition of the vertex set of the graph. This is tested in the remaining part of the algorithm. □

4 Triangle-free minimal strongly circular-imperfect graphs

By the Strong Perfect Graph Theorem, triangle-free minimal imperfect graphs are odd holes. We prove a similar result for strongly circular-perfectness: triangle-free strongly circular-imperfect graphs are some odd holes with at most 2 extra-vertices.

Require: a graph G
Ensure: boolean true if and only if G is triangle-free circular-perfect.

if G is bipartite **then**
  return TRUE
end if
compute a shortest odd cycle $\mathcal{O} = (o_1, \dots, o_{2p+1})$. (Note that a triangle is an odd cycle and that computing a shortest odd cycle is much easier than finding out a shortest odd hole).
From now on, indices are modulo $2p + 1$.
if p=1 **then**
  return FALSE
end if
for $i \in 1 \dots 2p + 1$ **do**
  $B_i := \{v|\text{deg}(v) = 1, |vo_i \in E(G)\}$
  $A_i := \{v|vo_{i-1} \in E(G),\ vo_{i+1} \in E(G)\} \cup \{o_i\}$
end for
for $i \in 1 \dots 2p + 1$ **do**
  if ($|A_i| > 1$ and ($|A_{i+1}| > 1$ or $|A_{i-1}| > 1$)) or ($B_i \neq \emptyset$ and $|A_i| > 1$) **then**
    return FALSE
  end if
end for
$V := \emptyset$; $E := \emptyset$
for $i \in 1 \dots 2p + 1$ **do**
  $V := V \cup A_i \cup B_i$
  $E_i := A_i \times A_{i+1}$; $E'_i := A_i \times B_i$; $E := E \cup E_i \cup E'_i$
end for
if $V \neq V(G)$ or $E \neq E(G)$ **then**
  return FALSE
end if
return TRUE

Algorithm 1: A polynomial time recognition algorithm for triangle-free strongly circular-perfect graphs

To be more precise, let us say that a graph G is an *extended odd hole* if it admits a proper partition into an induced odd hole $\mathcal{O} = \{o_1, \dots, o_{2p+1}\}$ and a pair of vertices $\{x, y\}$ which is connected to $\mathcal{O}$ in one of the following ways:

(a) $\{o_1x, xy, o_4y\}$
(b) $\{o_1x, xy, o_2y\}$
(c) $\{o_1x, o_3x, xy, o_4y\}$
(d) $\{o_1x, o_3x, xy, o_2y\}$
(e) $\{o_1x, o_3x, xy, o_2y, o_4y\}$
(f) $\{o_1x, o_3x, o_2y, o_4y\}$

Theorem 22 *A triangle-free graph G is minimal strongly circular-imperfect if and* *only if G is either the disjoint union of an odd hole and a singleton or an extended odd hole.*

Proof. ***Only if.*** Let G be a triangle-free minimal strongly circular-imperfect graph. If G does not have any induced odd hole then G is perfect, a contradiction. Let $\mathcal{O}$ be a shortest induced odd hole of G. Notice that $\mathcal{O} \subsetneq G$. Let $\{o_1, \ldots, o_{2p+1}\}$ be a labeling of the vertices of $\mathcal{O}$ the usual way ($o_i o_{i+1}$ is an edge of $\mathcal{O}$ for every $1 \leq i \leq 2p+1$, and indices modulo $2p+1$).

Claim 23 *If there is a vertex x of degree 0 then G is the disjoint union of $\mathcal{O}$ and the singleton x.*

If x is of degree 0 then $x \notin \mathcal{O}$. By Lemma 11, the induced subgraph $\mathcal{O} \cup \{x\}$ is strongly circular-imperfect, hence $G = \mathcal{O} \cup \{x\}$. $\diamond$

Claim 24 *If there is a unique vertex x of G outside $\mathcal{O}$ then G is the disjoint union of $\mathcal{O}$ and the singleton x.*

If x is not isolated, G is an interlaced odd hole by Lemma 13, a contradiction. $\diamond$

Thus, we may assume from now on, that G has at least two vertices outside $\mathcal{O}$. We have to prove that G is an extended odd hole.

Claim 25 *Every vertex of G is of degree at least 2.*

By Claim 23, every vertex is of degree at least 1. If there exists a vertex v in G of degree 1, then obviously $v \notin \mathcal{O}$. Notice that $G' = G - v$ is triangle-free strongly circular-perfect. Hence by Theorem 15, G' is bipartite or an interlaced odd hole. The case G' bipartite is excluded, otherwise G would be also bipartite. Let $((A_i)_{i=1..2p+1}, (B_i)_{i=1..2p+1})$ be a suitable partition of G'. The neighbour w of v belongs obviously to $\mathcal{O}$ (if not, $\mathcal{O} \cup \{v\}$ would be a proper induced strongly circular-imperfect graph). Thus there exists an index i such that $w \in A_i$. If A_i is of size 1 then G is an interlaced odd hole, a contradiction. Hence there exists $t \in A_i \setminus \{w\}$. Thus $((\mathcal{O} \setminus \{w\}) \cup \{t\}) \cup \{v\}$ is a proper induced subgraph of G which is the disjoint union of an odd hole and a singleton, and is therefore strongly circular-imperfect, a final contradiction. $\diamond$

Claim 26 *If G has at least 3 vertices outside $\mathcal{O}$ then $G \setminus \mathcal{O}$ is a stable set and for every vertex v of G outside $\mathcal{O}$, there exists an index $f(v)$ such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p+1$).*

Assume that there is an edge ab which is not incident to $\mathcal{O}$ and let c be a third vertex outside $\mathcal{O}$. Then $G - c$ is an interlaced odd hole with the edge ab which is not incident to the odd hole $\mathcal{O}$, a contradiction to Lemma 14. Hence $G \setminus \mathcal{O}$ is a stable set. Let v be a vertex of G outside $\mathcal{O}$. Let w be another vertex of G outside $\mathcal{O}$. Since $G - w$ is an interlaced odd hole and $v \notin \mathcal{O}$, this implies with Claim 25 that v has exactly two neighbours on $\mathcal{O}$, and that there exists an index $f(v)$ such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p + 1$). $\diamond$

Claim 27 *There are exactly two vertices of G outside $\mathcal{O}$.*

Assume that there are at least 3 vertices outside $\mathcal{O}$. Hence Claim 26 applies: for every $v \notin \mathcal{O}$, let $f(v)$ be the index such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p + 1$).

For every $1 \leq i \leq 2p + 1$, let A_i be the set of vertices $\{o_i\} \cup \{v| \, v \notin \mathcal{O}, \, f(v) = i - 1\}$. Notice that the edge set of G is precisely $\cup_{i=1,\ldots,2p+1} E_i$, where E_i denotes the set of all edges between A_i and A_{i+1}. Every set A_i is obviously non-empty. If there exists i such that A_i and A_{i+1} are both of size at least 2, then let $a_i \in A_i \setminus \{o_i\}$ and let $a_{i+1} \in A_{i+1} \setminus \{o_{i+1}\}$. Since there are at least 3 vertices outside $\mathcal{O}$, there is also a vertex z outside $\mathcal{O}$, distinct of a_i and a_{i+1}. Then $G - z$ is an interlaced odd hole, with a shortest odd hole $\mathcal{O}' = (\mathcal{O} \setminus \{o_i\}) \cup \{a_i\}$ and an edge $o_i a_{i+1}$ which is not incident to $\mathcal{O}'$: a contradiction with Lemma 14. Hence $\forall 1 \leq i \leq 2p + 1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (with indices modulo $2p + 1$).

Therefore G is an interlaced odd hole and is circular-perfect: a contradiction. Hence there are exactly two 2 vertices outside $\mathcal{O}$. $\diamond$

From now on, assume that x and y are the two distinct vertices of G outside $\mathcal{O}$. Since $G - y$ (resp. $G - x$) is an interlaced odd hole and $x \notin \mathcal{O}$ (resp. $y \notin \mathcal{O}$), this implies that there exists an index $f(x)$ (resp. $f(y)$) such that $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ or $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$ (resp. $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ or $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$) (with indices modulo $2p + 1$).

Claim 28 *If x is not adjacent to y then G is an extended odd hole of type f.*

Due to Claim 25, we have $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$. Notice that if $f(x) \neq f(y) \pm 1 \pmod{2p + 1}$ then G is an interlaced odd hole, a contradiction. Hence $f(x) = f(y) \pm 1 \pmod{2p + 1}$ and G is an extended odd hole of type f. $\diamond$

In the following, we assume that x is adjacent to y. We have to prove that G is an extended odd hole of type a, b, c, d or e.

Claim 29 *If $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ then G is an extended odd hole of type e.*

Let $z = o_{f(y)+1}$. Notice that $\mathcal{O}' = (\mathcal{O} \setminus \{z\}) \cup \{y\}$ is an induced odd hole of $G - z$. If $z \neq o_{f(x)}$ or $o_{f(x)+2}$ then x is a vertex of $G - z$ outside $\mathcal{O}'$ with 3 neighbours in $\mathcal{O}'$. Hence $G - z$ is not an interlaced odd hole, a contradiction as it is not bipartite. Thus $f(x) = f(y) \pm 1$ and G is an extended odd hole of type e. $\diamond$

Claim 30 *If ($N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$) or ($N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ and $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$) then G is an extended odd hole of type c or d.*

Assume w.l.o.g. that $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$. Let $z = o_{f(x)+1}$. Notice that $\mathcal{O}' = (\mathcal{O} \setminus \{z\}) \cup \{x\}$ is an induced odd hole of $G - z$. If $z = o_{f(y)}$ then G is an extended odd hole of type d. If $z \neq f(y)$ then y has two neighbours in $\mathcal{O}'$, and one of them is x. Since $G - z$ is an interlaced odd hole, this implies that $o_{f(y)}$ is at distance 2 in $\mathcal{O}'$ from x. Hence $f(y) = f(x) + 3$ of $f(y) = f(x) - 1$. In both cases, G is an extended odd hole of type c. $\diamond$

Claim 31 *If $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$ then G is an extended odd hole of type a or b.*

Assume w.l.o.g. that $f(x) \geq f(y)$. The case $f(x) = f(y)$ is excluded as G is triangle-free. If $f(x) - f(y)$ is even, notice that $\{x, y, o_{f(y)}, o_{f(y)+1}, \ldots, o_{f(x)}\}$ induces an odd hole. If $f(x) = f(y) + 2p$ then $f(x) = 1$, $f(y) = 2p + 1$ and G is an extended odd hole of type b. If $f(x) < f(y) + 2p$ then the subgraph $G \setminus \{o_{f(x)+1}\}$ is an interlaced odd hole. Hence $f(x) + 2$ is adjacent to the odd hole $\{x, y, f(y), f(y) + 1, \ldots, f(x)\}$. Thus $(f(x) + 2) + 1 = f(y) \pmod{2p + 1}$. This implies that G is an extended odd hole of type a as $f(y) = f(x) + 3 \pmod{2p + 1}$. If $f(x) - f(y)$ is odd, notice that $\{x, y\} \cup \{1, 2 \ldots, o_{f(y)}\} \cup \{o_{f(x)}, o_{f(x)+1}, \ldots, 2p+1\}$ induces an odd hole. If $f(x) = f(y)+1$ then G is an extended odd hole of type b. If $f(x) > f(y) + 1$ then the subgraph $G \setminus \{o_{f(y)+1}\}$ is an interlaced odd hole. Hence $o_{f(y)+2}$ is adjacent to the odd hole $\{x, y\} \cup \{1, 2 \ldots, o_{f(y)}\} \cup \{o_{f(x)}, o_{f(x)+1}, \ldots, 2p+1\}$. Thus $(f(y)+2)+1 = f(x) \pmod{2p + 1}$. This implies that G is an extended odd hole of type a as $f(x) = f(y) + 3 \pmod{2p + 1}$. $\diamond$

If. The disjoint union of an odd hole and a singleton is strongly circular-imperfect due to Lemma 11. If G is an extended odd hole, then G is strongly circular-imperfect as no extended odd hole is an interlaced odd hole. Let v be a vertex of G. It is straightforward to check that $G - v$ is bipartite or an interlaced odd hole, and therefore strongly circular-perfect, whatever the type (a, b, c, d, e or f) of G is. $\square$

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